PROTOTYPE EXAMPLE 1 - WYNDOR GLASS CO.

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4 × 6 foot double-hung wood-framed window

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix* of the two products would be *most profitable*. Therefore, an OR team has been formed to study this question.

The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

> Determine what the *production rates* should be for the two products in order to *maximize their total profit*, subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the *production rate* is defined as the number of batches produced per week.) *Any* combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
2. Number of hours of production time used in each plant for each batch produced of each new product.
3. Profit per batch produced of each new product. (*Profit per batch produced* was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly *constant* regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this *profit per batch produced* times *the number of batches produced*.)

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production processes for the new products. By analyzing cost data from these same engineers and the marketing division, along with a pricing decision from the marketing division, the accounting department developed estimates for the third category.

Table 3.1 summarizes the data gathered.

The OR team immediately recognized that this was a linear programming problem of the classic product mix type, and the team next undertook the formulation of the corresponding mathematical model.

Table 3.1

	Production Time per Batch, Hours		
	Product		Production Time Available per Week, Hours
Plant	1	2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

a) Formulate and solve the problem.
b) Consider the same set of feasible solutions and determine the set of optimal solutions if the objective is
b1) $Max\ Z = 5x_1 + x_2$; b2) $Max\ Z = 6x_1 + 4x_2$; b3) $Min\ Z = -x_1 + x_2$
b4) $Min\ Z = x_1 - x_2$; b5) $Max\ Z = x_1$; b6) $Max\ Z = 4x_2$;
c) Top management wants to know the consequences if a minimum of $50,000 of profit is required.
d) Solve the initial problem assuming that the capacity of Plants 2 and 3 is unlimited. Repeat b) with this new feasible region.
e) Consider that the total of 18 h.w. available in Plant 3 must be used. Keeping the remaining initial constraints, identify and explain what is the new feasible region and optimal solution.
f) Solve the initial problem considering that the amount of windows cannot be smaller than the quadruple of the amount of doors.

PROTOTYPE EXAMPLE 2 – Planning an Advertising Campaign

The Profit & Gambit Co. produces cleaning products for home use. This is a highly competitive market, and the company continually struggles to increase its small market share. Management has decided to undertake a major new advertising campaign that will focus on the following three key products:

- A spray prewash stain remover.
- A liquid laundry detergent.
- A powder laundry detergent.

This campaign will use both television and the print media. A commercial has been developed to run on national television that will feature the liquid detergent. The advertisement for the print media will promote all three products and will include cents-off coupons that consumers can use to purchase the products at reduced prices. The general goal is to increase the sales of each of these products (but especially the liquid detergent) over the next year by a significant percentage over the past year. Specifically, management has set the following goals for the campaign:

- Sales of the stain remover should increase by at least 3 percent.
- Sales of the liquid detergent should increase by at least 18 percent.
- Sales of the powder detergent should increase by at least 4 percent.

Table 2.2 shows the estimated increase in sales for each *unit* of advertising in the respective outlets.[4] (A *unit* is a standard block of advertising that Profit & Gambit commonly purchases, but other amounts also are allowed.) The reason for −1 percent for the powder detergent in the Television column is that the TV commercial featuring the new liquid detergent will take away some sales from the powder detergent. The bottom row of the table shows the cost per unit of advertising for each of the two outlets.

Management's objective is to determine how much to advertise in each medium to meet the sales goals at a minimum total cost.

Formulate and solve the problem (graphically)

Table 2.2

	Increase in Sales per Unit of Advertising		
Product	Television	Print Media	Minimum Required Increase
Stain remover	0%	1%	3%
Liquid detergent	3%	2%	18%
Powder detergent	-1%	4%	4%
Cost per unit	$1,000	$2,000	

1. Consider the following LP problems:

a) $Max\ Z = x_1 + 2x_2$

$$s.t. \begin{cases} x_1 - 2x_2 \leq 3 \\ x_1 + x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases}$$

b) $Max\ Z = 3x_1 + 4x_2$

$$s.t. \begin{cases} x_1 - 2x_2 \geq 4 \\ x_1 + x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases}$$

c) $Max\ Z = x_1 + x_2$

$$s.t. \begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

d) $Max\ Z = x_1 - x_2$

$$s.t. \begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

e) $Max\ Z = -10x_1 - 5x_2$

$$s.t. \begin{cases} x_1 - x_2 \leq 5 \\ x_1 + \frac{8}{5}x_2 \geq -3 \\ x_1\ free, x_2 \leq 0 \end{cases}$$

f) $Min\ Z = x_1 + x_2$

$$s.t. \begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

g) $Max\ Z = x_1 + x_2$

$$s.t. \begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

h) $Min\ Z = x_1 + x_2$

$$s.t. \begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq -2 \\ x_1, x_2 \geq 0 \end{cases}$$

i) $Min\ Z = 3x_1 + 2x_2$

$$s.t. \begin{cases} x_1 + x_2 \geq 3 \\ 3x_1 + 2x_2 \leq 18 \\ 5x_1 + 2x_2 = 10 \\ x_1, x_2 \geq 0 \end{cases}$$

j) $Max\ Z = 3x_1 + 6x_2$

$$s.t. \begin{cases} x_1 + 2x_2 \leq 4 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

1. Solve by the graphical method and by the Solver.
2. Write the dual and find the solution based on part 1 and on dual properties.
3. Solve by the simplex method problems a), c), d), e), h) and j).

2. Formulate a linear programming model for each of the following problems and solve it by the solver of Excel:

a) The Omega Manufacturing Company has discontinued the production of a certain unprofitable product line. This act created considerable excess production capacity. Management is considering devoting this excess capacity to one or more of three products; call them products 1, 2, and 3. The available capacity on the machines that might limit output is summarized in the following table:

Machine Type	Available Time (Machine Hours per Week)
Milling machine	500
Lathe	350
Grinder	150

The number of machine hours required for each unit of the respective products is

Productivity coefficient (in machine hours per unit)

Machine Type	Product 1	Product 2	Product 3
Milling machine	9	3	5
Lathe	5	4	0
Grinder	3	0	2

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 units per week. The unit profit would be \$50, \$20, and \$25, respectively, on products 1, 2, and 3. The objective is to determine how much of each product Omega should produce to maximize profit.

(a) Formulate a linear programming model for this problem.

C **(b)** Use a computer to solve this model by the simplex method.

b) A farm family owns 125 acres of land and has \$40,000 in funds available for investment. Its members can produce a total of 3,500 person-hours worth of labor during the winter months (mid-September to mid-May) and 4,000 person-hours during the summer. If any of these person-hours are not needed, younger members of the family will use them to work on a neighboring farm for \$5/hour during the winter months and \$6/hour during the summer.

Cash income may be obtained from three crops and two types of livestock: dairy cows and laying hens. No investment funds are needed for the crops. However, each cow will require an investment outlay of \$1,200, and each hen will cost \$9.

Each cow will require 1.5 acres of land, 100 person-hours of work during the winter months, and another 50 person-hours during the summer. Each cow will produce a net annual cash income of \$1,000 for the family. The corresponding figures for each hen are: no acreage, 0.6 person-hour during the winter, 0.3 more person-hour during the summer, and an annual net cash income of \$5. The chicken house can accommodate a maximum of 3,000 hens, and the size of the barn limits the herd to a maximum of 32 cows.

Estimated person-hours and income per acre planted in each of the three crops are

	Soybeans	Corn	Oats
Winter person-hours	20	35	10
Summer person-hours	50	75	40
Net annual cash income (\$)	600	900	450

The family wishes to determine how much acreage should be planted in each of the crops and how many cows and hens should be kept to maximize its net cash income. Formulate the linear programming model for this problem.

c) A farmer is raising pigs for market, and he wishes to determine the quantities of the available types of feed that should be given to each pig to meet certain nutritional requirements at a *minimum cost*. The number of units of each type of basic nutritional ingredient contained within a kilogram of each feed type is given in the following table, along with the daily nutritional requirements and feed costs:

Nutritional Ingredient	Kilogram of Corn	Kilogram of Tankage	Kilogram of Alfalfa	Minimum Daily Requirement
Carbohydrates	90	20	40	200
Protein	30	80	60	180
Vitamins	10	20	60	150
Cost (¢)	42	36	30	

Formulate the linear programming model for this problem.

d) A certain corporation has three branch plants with excess production capacity. All three plants have the capability for producing a certain new product, and management has decided to use some of the excess capacity in this way. This product can be made in three sizes—large, medium, and small—that yield a net unit profit of $420, $360, and $300, respectively. Plants 1, 2, and 3 have the excess capacity to produce 750, 900, and 450 units per day of this product, respectively, regardless of the size or combination of sizes involved.

The amount of available in-process storage space also imposes a limitation on the production rates of the new product. Plants 1, 2, and 3 have 13,000, 12,000, and 5,000 square feet of in-process storage space available for a day's production of this product. Each unit of the large, medium, and small sizes produced per day requires 20, 15, and 12 square feet, respectively.

Sales forecasts indicate that 900, 1,200, and 750 units of the large, medium, and small sizes, respectively, can be sold per day.

To maintain a uniform workload among the plants and to retain some flexibility, management has decided that the plants must use the same percentage of their excess capacity to produce the new product.

Management wishes to know how much of each of the sizes should be produced by each of the plants to maximize profit.

Formulate the linear programming model for this problem.

e) The Energetic Company needs to make plans for the energy systems for a new building.

The energy needs in the building fall into three categories: (1) electricity, (2) heating water, and (3) heating space in the building. The daily requirements for these three categories (all measured in the same units) are

Electricity	30 units
Water heating	20 units
Space heating	50 units

The three possible sources of energy to meet these needs are electricity, natural gas, and a solar heating unit that can be installed on the roof. The size of the roof limits the largest possible solar heater to 40 units, but there is no limit to the electricity and natural gas available. Electricity needs can be met only by purchasing electricity (at a cost of $50 per unit). Both other energy needs can be met by any source or combination of sources. The unit costs are

	Electricity	Natural Gas	Solar Heater
Water heating	$150	$110	$70
Space heating	$140	$100	$90

The objective is to minimize the total cost of meeting the energy needs.

f) A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic Feet)
Front	12	7,000
Center	18	9,000
Back	10	5,000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

Cargo	Weight (Tons)	Volume (Cubic Feet/Ton)	Profit ($/Ton)
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

Formulate the linear programming model for this problem.

g) Al Ferris has $60,000 that he wishes to invest now in order to use the accumulation for purchasing a retirement annuity in 5 years. After consulting with his financial adviser, he has been offered four types of fixed-income investments, which we will label as investments A, B, C, D.

Investments A and B are available at the beginning of each of the next 5 years (call them years 1 to 5). Each dollar invested in A at the beginning of a year returns $1.40 (a profit of $0.40) 2 years later (in time for immediate reinvestment). Each dollar invested in B at the beginning of a year returns $1.70 three years later.

Investments C and D will each be available at one time in the future. Each dollar invested in C at the beginning of year 2 returns $1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns $1.30 at the end of year 5.

Al wishes to know which investment plan maximizes the amount of money that can be accumulated by the beginning of year 6.

h) This is your lucky day. You have just won a $10,000 prize. You are setting aside $4,000 for taxes and partying expenses, but you have decided to invest the other $6,000. Upon hearing this news, two different friends have offered you an opportunity to become a partner in two different entrepreneurial ventures, one planned by each friend. In both cases, this investment would involve expending some of your time next summer as well as putting up cash. Becoming a *full* partner in the first friend's venture would require an investment of $5,000 and 400 hours, and your estimated profit (ignoring the value of your time) would be $4,500. The corresponding figures for the second friend's venture are $4,000 and 500 hours, with an estimated profit to you of $4,500. However, both friends are flexible and would allow you to come in at any *fraction* of a full partnership you would like. If you choose a fraction of a full partnership, all the above figures given for a full partnership (money investment, time investment, and your profit) would be multiplied by this same fraction.

Because you were looking for an interesting summer job anyway (maximum of 600 hours), you have decided to participate in one or both friends' ventures in whichever combination would maximize your total estimated profit. You now need to solve the problem of finding the best combination.

(a) Describe the analogy between this problem and the Wyndor Glass Co. problem discussed in Sec. 3.1. Then construct and fill in a table like Table 3.1 for this problem, identifying both the activities and the resources.

(b) Formulate a linear programming model for this problem.

D,I (c) Use the graphical method to solve this model. What is your total estimated profit?

i) A company has at your service 100 skilled workers, semi-skilled 230 workers and 80 unskilled workers. The semi-skilled workers can be skilled workers if they attend a one year, these training courses cost 500 m.u. per worker. Unskilled workers can improve to be semi-skilled by training courses that 500 m.u. per worker. The company intends to plan training of its staff over the next two years so that at the end of the planning period:

1. unskilled workers can not represent more than 10% of the total;
2. at least 40% of the workers should attend a training course;
3. at least 35% of the amount spent in training should be on unskilled workers

How much should the company spent in training courses?

j) A plant imports three types of thread - cotton, wool and e fiber - to produce three different types of cloth - **C1**, **C2** and **C3**. The cloths should follow the specifications below:

	cloth	Selling price (m.u./kg)
C1	At least 60% of cotton & at most 20% of fiber	680
C2	at most 60% of fiber & At least 15% of wool	570
C3	at most 50% of fiber	450

The aim is to determine the production plan so that the profit is maximized, using the information about availabilities and cost given in the following table:

thread	availabilities (kg)	cost (m.u./kg)
Cotton	2 000	700
Wool	2 500	500
fiber	1 200	400

3. Solve the following problems by the simplex algorithm.

a) $\text{Max } Z = 2x_1 + 3x_2 + 2x_3$

$$\text{s.t.:} \begin{cases} x_1 + 2x_2 + 3x_3 \le 6 \\ x_1 + x_2 + x_3 \le 10 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

b) $\text{Min } Z = -3x_1 + x_2 - 2x_3$

$$\text{s.t.:} \begin{cases} x_1 + x_2 \le 3 \\ x_1 + 2x_2 + 2x_3 \le 6 \\ 2x_1 + 2x_2 + x_3 \le 8 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

c) $\text{Max } Z = 3x_1 + 4x_2 + x_3$

$$\text{s.t.:} \begin{cases} x_1 + 2x_2 + x_3 \le 5 \\ 2x_1 + 3x_2 + x_3 \le 10 \\ 3x_1 + x_2 + x_3 \le 8 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

d) $\text{Max } Z = 4x_1 + 5x_2 + 3x_3$

$$\text{s.t.:} \begin{cases} 7x_1 + 4x_2 + x_3 \le 10 \\ 2x_1 + 3x_2 + 2x_3 \le 4 \\ 3x_1 + 4x_2 + x_3 \le 11 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

4. A firm has a plant with the capacity to work 70 hours a week and can produce three products (P1, P2 e P3). Each unit of P3 requires one hour of that capacity, while the unit production of P1 and P2 needs, respectively, the double and the triple of that time. The three products, when finished, are stored in a warehouse with 100 m^3 available. Each unit of product (P1, P2 or P3) requires1 m^3. The gross unit margin achieved by each product is 10 (P1), 15 (P2) and 5 (P3).

a) Formulate an LP problem to maximize the total gross margin.
b) Find all the optimal solutions by simplex algorithm.

5. Consider the following LP problem P:

Max $Z = x_1 - 3x_2$

$$\text{s.t.:} \begin{cases} \frac{1}{3}x_1 + x_2 \leq 8 \\ x_1 - x_2 \leq 8 \\ x_1 \geq 0 \end{cases}$$

a) Solve P by the graphical method.

b) Determine the solution with the first constraint binding and $x_2 = 0$. Classify it.

c) Write P in the *standard* form and in the augmented form.

d) Solve P by the simplex algorithm.

6. Consider the following simplex tableau of a maximization problem:

BV	coefficients of Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	c	0	2	0	0	9
x_2	0	-1	1	a_1	0	0	3
x_4	0	a_2	0	-3	1	0	1
x_5	0	a_3	0	4	0	1	2

Determine the values of a_1, a_2, a_3 e c that make the following sentences true:

a) The current solution is optimal;

b) The current solution is optimal and at least one more basic alternative solution exists;

c) The objective function is unbounded.

7. In factory Choco three new types of chocolate bars are going to be made for the food industry. Each bar is made of sugar and chocolate only.

Bar	quantity of sugar (*kg/bar*)	Quantity of chocolate (*kg/bar*)	Profit of each chocolate bar (*m.u.*)
Type 1	*1*	*2*	*3*
Type 2	*1*	*3*	*7*
Type 3	*1*	*1*	*5*
availabilities (*kg*)	*50*	*100*	

To formulate the problem we define variables x_j, representing the number of chocolate bars **Type j** to make, where $j = 1,2,3$.

Answer to the following questions using, when needed, the *Solver/Excel* to find the solution of the LP problems.

a) For which unit profit values of chocolate bars of **Type 2** does the current solution remains optimal? Which will be the optimal solution in case the unit profit is *13 m.u.*?

b) Is it worth considering an increase in the availability of sugar?

c) For which amount of sugar is the set of basic variables in the optimal solution the same?

d) Is it worth considering an increase in the availability of chocolate?

e) For which amount of chocolate is the set of basic variables in the optimal solution the same?

f) If the amount of sugar available was of *60 kg*, which would be the total profit of these products? Which should be the production plan that Choco should apply in these conditions?

g) Repeat the previous question for an availability of sugar of *40 kg* and *30 kg*.

8. An individual, among many others, strongly invested in real estate based funds, and finally managed to recover 25 thousand *m.u.* (monetary units) which he intends to invest during a certain period of time. After the past experience, his goal is to minimize the risk, however he would like to achieve a minimum return of 2 thousand *m.u.* at the end of the time period. The characteristics of the financial products, which he ponders to include in his portfolio, made him formulate such LP problem, where x_i represents the amount (in 10^3 *m.u.*) to be invested on product i=1,2:

$$Min\, z = x_1 + 2x_2$$

$$\text{s.to} \begin{cases} x_1 + x_2 \leq 25 & \text{(budget constraint)} \\ 0{,}5x_1 + 0{,}8x_2 \geq 2 & \text{(return constraint)} \\ x_1,\ x_2 \geq 0 \end{cases}$$

a) Solve graphically the given problem. Present and interpret the optimum value of the decision and slack (auxiliary) variables.

b) Write the dual and determine its optimal solution (only the decision variables). Note that you can take advantage on the solution and resolution of a).

9. An account manager criticized the approach of the above problem, arguing that this way the return achieved would never be higher than the minimum required. According to him, the objective function should translate the maximization of the return and the risk might be controlled by constraints imposed on the portfolio composition. Besides, he suggested two additional financial products to take into account. Using the OR knowledge he gathered the following information.

	X1	X2	X3	X4	total		RHS (thousand *m.u.*)
Budget	1	1	1	1	25	<=	25
Risk 1	1	1	0	0	10	<=	15
Risk 2	0	0	1	1	15	<=	15
Risk 3	1	0	1	0	15	>=	15
Return	0,50	0,80	0,75	0,90	19,25		

a) Give the LP formulation and solve by S*olver/Excel*.

b) How much should be invested on each product and what is the associated total return?

c) How much does the total return change, if Risk 2 constraint is changed allowing a maximum of 14 thousand *m.u.* to invest, instead of the current 15 thousand *m.u.*?

d) Could you quantify the change in the total return, if it is required that the total invested in products 3 and 4 (RHS of constraint Risk 2) does not surpass 9 thousand *m.u.*?

e) How much does the total return change, if the return of product 1 increases from 0,5 to 0,6? Identify the optimal solution for this situation.

10. Consider the following LP problem:

$$MaxZ = 3x_1 + 2x_2 \qquad \text{(total profit, in } m.u.\text{)}$$

$$\text{s.t.:} \begin{cases} x_1 \leq 4 & \text{(resource 1)} \\ x_1 + 3x_2 \leq 15 & \text{(resource 2)} \\ 2x_1 + x_2 \leq 10 & \text{(resource 3)} \\ x_1, x_2 \geq 0 \end{cases}$$

a) Solve the problem by the graphical method, by the Simplex algorithm and by the *Solver/Excel*.

b) Write the dual of the above formulated problem.

c) Find the optimal solution of the dual with the help of the graphical solution of the primal, reading the output reports of *Solver* and solving the dual itself by the *Solver*.

d) Assuming that it is a problem to find the level at which activities that share limited resources should be performed, explain the economic meaning of the optimal solutions.

e) Determine the impact in the total profit of a reduction of the availability of resource 3 to 8 units.

11. Write the dual associated to the following LP problem:

$$MaxZ = 6x_1 + 8x_2$$

$$\text{s.a:} \begin{cases} 5x_1 + 2x_2 \leq 20 \\ x_1 + 2x_2 \leq 10 \\ x_1, x_2 \geq 0 \end{cases}$$

a) Solve by the graphical method the pair of dual problems.

b) Solve the primal problem by an algorithm. Write and classify the solution associated to each simplex tableau and identify them in the graphic.

12. Consider the following LP problem:

$$MaxZ = 2x_1 + 7x_2 + 4x_3$$

$$\text{s.t.:} \begin{cases} x_1 + 2x_2 + x_3 \leq 10 \\ 3x_1 + 3x_2 + 2x_3 \leq 10 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

a) Write its dual.

b) Solve both problems.

13. Consider the following LP problem::

$$MinZ = x_1 + 3x_2$$

$$\text{s.t.:} \begin{cases} x_1 + x_2 \geq 4 \\ -x_1 + x_2 \geq 0 \\ -x_2 \geq -6 \\ x_1,\ x_2 \geq 0 \end{cases}$$

a) Solve it by the graphical method and write the dual problem associated with this LP problem.

b) Solve the given problem by *Solver/Excel* and display the solutions of both problems.

14. A firm wants to study the future production plan of products **P1**, **P2** and **P3**. In order to maximize the global profit the following LP problem was formulated:

$$MaxZ = 3x_1 + 4x_2 + 2x_3$$

$$\text{s.t.:} \begin{cases} x_1 + x_2 + 2x_3 \leq 10 \\ 2x_1 + 4x_2 + x_3 \leq 8 \\ 2x_1 + 3x_2 + 2x_3 \leq 20 \\ x_1,\ x_2,\ x_3 \geq 0 \end{cases}$$

Where x_j represents the quantity of product **Pj**, $j = 1, 2, 3$ that should be produced;

the first two constraints refer to the consumption of raw materials **rm1** and **rm2**, respectively; the third constraint is associated to the limited availability of store space in the warehouse.

Assume that the optimal production plan indicates that only *2* units of **P1** and *4* units of **P3** should be produced.

a) Without solving the problem but with the information that the first shadow-price is *1/3*, determine the internal values of the resources (raw materials and store space) and give the economic interpretation of those values.

b) Obtain the output *Solver/Excel* reports and find the increase in the actual unit profit of **P2** in order that it becomes advantageous to include it in the production plan.

15. Consider the following LP problem:

$$MinZ = 3x_1 + 2x_2$$

$$\text{s.t:} \begin{cases} x_1 \leq 3 \\ 3x_2 \leq 12 \\ \alpha x_1 + x_2 \geq 6 \\ x_1,\ x_2 \geq 0 \end{cases}$$

a) Take $\alpha = 1$ and solve the problem. Write the dual and solve it.

b) Find a value for α so that alternative optimal solutions can be found. Explain.

c) Find a value for α so that the problem is infeasible.

16. Consider the following LP problem, formulated by the OR department of a company, which intends to optimize the total monthly revenue from the sale of four products (**P1**, **P2**, **P3** and **P4**), produced using two types of raw material (**rm1** and **rm2**):

$$MaxZ = 2x_1 + 3x_2 + 7x_3 + 4x_4$$

$$\text{s.t.:}\begin{cases} x_1 + x_2 + x_3 + x_4 \leq 9 & \textbf{(rm1)} \\ x_1 + 2x_2 + 4x_3 + 8x_4 \leq 24 & \textbf{(rm2)} \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Considering the solution of the problem, the department informs that only products **P1** and **P3** should be produced.

a) Which of the two products not included in the production plan, would need a smaller increase in the unit selling price in order that its production turns to be profitable?

b) Assume that due to difficulties on import, next month only *5* units of **rm1** will be available. Determine the new optimal production plan and the total revenue associated to it.

c) Suppose that the company can produce a new product, with a unit selling price of *10 m.u.*, and a need of *2* units from each one of the raw materials to produce one unit of the new product. What should be the new production plan?

d) Suppose that a budget of *24 m.u.* is now available to spend in only one of the raw materials. Let *4m.u.* and *8 m.u.* the unit price to acquire each extra unit of that raw materials, respectively. Which decision should be made if the companies' management pretends keep the production of **P1** and **P3** only? What are the consequences for the company?

17. Consider the *output* of *Solver/Excel* of the following problem, where x_1, x_2 and x_3 are the quantities sold of products 1, 2 and 3, respectively:

$$Min\ Z = 2x_1 + 3x_2 + 8x_3 \quad \text{(total cost)}$$

$$\text{s.t.:}\begin{cases} x_1 + x_2 + x_3 \geq 90 & \text{(minimum quantity)} \\ 5x_1 + 4x_2 + 3x_3 \leq 400 & \text{(maximum capacity)} \\ 3x_1 - 5x_2 = 0 & \text{(relation between sells of products 1 and 2)} \\ x_1,\ x_2, x_3 \geq 0 \end{cases}$$

a) Display the optimal solution of the primal and interpret the meaning of all its variables (including slack variables).

b) Display and interpret the shadow prices.

c) What are the consequences in the total cost if the third constraint changes to $3x_1 - 5x_2 = 100$?

d) Suppose that the unit cost of product 3 is now *4.5*, determine the consequences in the selling plan and in the total cost.

18. A humanitarian organization intends to plan a medicaments distribution program in two regions located in the Great Lakes area of Africa. For strategic and security purposes it is possible to use 3 airports from which, by land routes, the supply of the two regions will take place. Considering that the transportation cost of medicaments to the airports should be minimized, insuring, in each of the two regions, a minimum number of people is contemplated by the program, the following LP model has been formulated:

$$MinZ = 40x_1 + 18x_2 + 30x_3 \qquad \text{(in m.u.)}$$

$$\text{s.a}: \begin{cases} 4x_1 + x_2 + x_3 \geq 250 & \text{(thounsands of people)} \\ 4x_1 + 3x_2 + 6x_3 \geq 350 & \text{(thounsands of people)} \\ x_1,\, x_2,\, x_3 \geq 0 \end{cases}$$

where $x_j =$ tones of medicaments to be shipped to airport j ($j = 1,2,3$).

a) Obtain the optimal solution for the problem by *Solver/Excel*.

b) Write short report presenting the problem's solution, referring the value of the dual decision variables as well as its meaning.

c) If the number of thousands of people to be contemplated in the first region is *350* instead of *250*, which will be the new cost for the program?

d) Determine the changes in the solution if airport 1 cannot receive more than *40 ton.* of medicaments.

19. Solve the following LP problem referring to the production of **P1**, **P2** e **P3,** using *Solver/Excel*:

$$MaxZ = 10\,x_1 + 20\,x_2 + 15\,x_3$$

$$\text{s.a:} \begin{cases} x_1 + 3\,x_2 + 2x_3 \leq 80 \\ 4\,x_1 + 10\,x_2 + 5x_3 \leq 90 \\ 4\,x_1 + 10\,x_2 \geq 50 \\ x_1,\, x_2,\, x_3 \geq 0 \end{cases}$$

The objective function refers to the maximization of the total revenue, and the first and the second constraints are related to the machine hours available in sections 1 and 2, respectively, and the third constraint determines the minimum financial margin that should be achieved. The financial margin is the difference between the total revenue and the total variable costs.

a) Write and interpret the optimal solutions for the primal and dual problems.

b) The revenue of **P2** just increased *20%*, although its financial margin is maintained. What are the changes on the production program and the shadow-prices.

c) How much should the revenue of **P2** increase (maintaining its financial margin) so that this product is included in the production plan.

d) Indicate a way of increasing the revenue by at least *2%* through the changes in the available quantities of the company´s resources.

e) Which are the changes in the optimal solution arising from the new market requirements which obligates a minimum production for **P2** of *4* units? Same question for **P1**.

20. Consider the following LP problem:

$$Min\ Z = x_1$$

$$\text{s.t:} \begin{cases} 2x_1 + 2x_2 \geq 10 \\ -x_1 + x_2 \leq 5 \\ 3x_1 + 2x_2 \leq 30 \\ 2x_1 - x_2 \leq 16 \\ x_1, x_2 \geq 0 \end{cases}$$

a) Solve it by graphical method.

b) Perform the sensitivity analysis to the right hand side of the third functional constraint graphically.

c) Without solving the dual, determine the optimal value of the dual variable associated to the third functional constraint.

d) Without using the graphic (nor solving the new problem) check if the introduction of the constraint $x_1 + x_2 \geq 3$ changes the feasible region.